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SPONTANEOUS SWIRLING IN AXISYMMETRIC FLOWS OF A CONDUCTING FLUID IN A MAGNETIC FIELD

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The problem of spontaneous swirling was considered in [1–7] and is as follows: can rotary motion occur in the absence of external source of rotation, i.e., under conditions where motion without rotation is realizable?

A more rigorous formulation of this problem was given by Lugovtsov [7]. The proposed formulation ensures a close control of the kinematical flux of the axial component of the angular momentum, which eliminates inflow of the rotating fluid in the flow region.

The occurrence of rotary motion is regarded as a bifurcation of the initial axisymmetric flow due to the loss of stability against swirling flow (not necessarily rotationally symmetric).

At present, examples of the occurrence of spontaneous swirling [1–3], including swirling in MHD flow [5, 6], have been given. However, as was shown in [4, 7], the available examples do not satisfy the more rigorous requirements formulated in [7]. Thus, the question of the possibility of spontaneous swirling remains open.

The proof that spontaneous swirling is impossible, if this statement is valid, involves significant difficulties and can hardly be obtained in a fairly general form. To prove the existence of this phenomenon, it is sufficient to find at least one example. To narrow the region of search for such an example, it is of interest to consider transition of axisymmetric to rotationally symmetric flow or a plane analog of this transition, i.e., the occurrence of a spontaneous cross (normal) flow which is independent of the transverse coordinate in the case of an initial plane-parallel flow [7].

Lugovtsov [7] showed that the bifurcation axisymmetric flow–rotationally symmetric flow (and the corresponding plane analog of this transition) does not take place for a compressible fluid with a variable viscosity coefficient. In the case of the plane analog, this statement is also valid for a conductive fluid moving in the presence of a magnetic field, irrespective of the character of connectedness of the flow region.

Such a general result is difficult to obtain for axisymmetric flows in the presence of a magnetic field. In this case, as was noted by Lugovtsov and Gubarev [7], swirling flows can occur which are maintained by electromagnetic forces, and the formulation of the problem of spontaneous swirling requires refinement.

Below, we consider axisymmetric flow of an incompressible conductive fluid. It will be shown that axisymmetric spontaneous swirling is impossible if the meridional section of the flow region is simply connected.

The equations describing such flows have the following form in conventional notation:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \nu\Delta\mathbf{v} - \frac{1}{4\pi\rho}(\mathbf{H}\times \mathrm{rot}\,\mathbf{H}) + \mathbf{f}, \quad \mathrm{div}\,\mathbf{v} = 0; \tag{1}$$

$$\mathrm{rot}\,\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}, \quad \mathrm{rot}\,\mathbf{H} = \frac{4\pi\sigma}{c}\Big[\mathbf{E} + \frac{1}{c}(\mathbf{v}\times\mathbf{H})\Big]; \tag{2}$$

$$\mathrm{div}\,\mathbf{E} = 4\pi\rho_e, \quad \mathrm{div}\,\mathbf{H} = 0. \tag{3}$$

Here $\mathbf{f} = (f_r(r,z,t), 0, f_z(r,z,t))$ are forces that maintain the initial axisymmetric flow; $\mathbf{v} = (u(r,z,t), 0, w(r,z,t))$.

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Using the scalar Φ and vector $\mathbf{A}$ electromagnetic potentials, we write Eqs. (2) and (3) as

$$\mathbf{H} = \text{rot } \mathbf{A}, \quad \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\Phi, \quad \text{div } \mathbf{A} = 0; \tag{4}$$

$$\frac{\partial \mathbf{A}}{\partial t} - \mathbf{v} \times \mathbf{H} = -\nu_m \text{ rot } \mathbf{H} - c\nabla\Phi, \quad \nu_m = \frac{c^2}{4\pi\sigma}; \tag{5}$$

$$\Delta\Phi = -4\pi\rho_e. \tag{6}$$

For the azimuthal component of the vector potential A_φ, taking into account axial symmetry, from (5) we obtain

$$\frac{\partial A_\varphi}{\partial t} + \frac{u}{r}\frac{\partial}{\partial r}(rA_\varphi) + w\frac{\partial A_\varphi}{\partial z} = \nu_m\Big[\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(rA_\varphi) + \frac{\partial^2 A_\varphi}{\partial z^2}\Big]. \tag{7}$$

For the poloidal components of the magnetic field H_r and H_z, we have

$$H_r = -\frac{\partial A_\varphi}{\partial z}, \quad H_z = \frac{1}{r}\frac{\partial}{\partial r}(rA_\varphi). \tag{8}$$

We introduce the "magnetic stream function" Ψ in place of A_φ, and assuming that $\Psi = rA_\varphi/(4\pi\rho)^{1/2}$, according to (8), we have

$$h_r = -\frac{1}{r}\frac{\partial \Psi}{\partial z}, \quad h_z = \frac{1}{r}\frac{\partial \Psi}{\partial r}, \quad \mathbf{h} = \frac{\mathbf{H}}{(4\pi\rho)^{1/2}}. \tag{9}$$

From (7), for Ψ we obtain

$$\frac{\partial \Psi}{\partial t} + u\frac{\partial \Psi}{\partial r} + w\frac{\partial \Psi}{\partial z} = \nu_m\Big(\frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r}\frac{\partial \Psi}{\partial r} + \frac{\partial^2 \Psi}{\partial z^2}\Big). \tag{10}$$

The equations for the azimuthal components of velocity $v_\varphi = v$ and of magnetic field $h_\varphi = h$ have the form

$$\frac{\partial \Gamma}{\partial t} + u\frac{\partial \Gamma}{\partial r} + w\frac{\partial \Gamma}{\partial z} = \nu\Big(\frac{\partial^2 \Gamma}{\partial r^2} - \frac{1}{r}\frac{\partial \Gamma}{\partial r} + \frac{\partial^2 \Gamma}{\partial z^2}\Big) - \frac{1}{r}\Big(\frac{\partial \Psi}{\partial z}\frac{\partial \gamma}{\partial r} - \frac{\partial \Psi}{\partial r}\frac{\partial \gamma}{\partial z}\Big); \tag{11}$$

$$\frac{\partial \gamma}{\partial t} + u\frac{\partial \gamma}{\partial r} + w\frac{\partial \gamma}{\partial z} - \frac{2u}{r}\gamma = \nu_m\Big(\frac{\partial^2 \gamma}{\partial r^2} - \frac{1}{r}\frac{\partial \gamma}{\partial r} + \frac{\partial^2 \gamma}{\partial z^2}\Big) - \frac{1}{r}\Big(\frac{\partial \Psi}{\partial z}\frac{\partial \Gamma}{\partial r} - \frac{\partial \Psi}{\partial r}\frac{\partial \Gamma}{\partial z}\Big) + \frac{2\Gamma}{r^2}\frac{\partial \Psi}{\partial z}, \tag{12}$$

where $\Gamma = rv$ and $\gamma = rh$.

Thus, system (10)–(12), taking into account relations (9), describes the behavior of the magnetic field and the azimuthal velocity component. The poloidal velocity components $u(r, z, t)$ and $w(r, z, t)$ satisfy the condition of incompressibility $\text{div}\,\mathbf{v} = 0$ and do not have singularities in the flow region (the radial velocity component on the axis of symmetry vanishes if the axis of symmetry belongs to the flow region). Otherwise the poloidal velocity components are arbitrary.

In what follows, it is assumed that the boundaries (walls) of the axisymmetric region are impenetrable and superconducting, so that on the boundary (on the boundaries, if the meridional section is not simply connected) the nonpenetration condition $\mathbf{vn} = 0$ and the condition $\mathbf{hn} = 0$ are satisfied. In addition, it is necessary to satisfy the condition of vanishing of the tangent to the walls of the electric-field component. From (8) follows

$$cE_r = -\nu_m\frac{\partial H_\varphi}{\partial z} - vH_z + wH_\varphi; \tag{13}$$

$$cE_\varphi = -\nu_m\Big(\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r}\Big) - wH_r + uH_z; \tag{14}$$

$$cE_z = \frac{\nu_m}{r}\frac{\partial}{\partial r}(rH_\varphi) - uH_\varphi + wH_r. \tag{15}$$

Equality of the tangential component of E to zero is ensured by the conditions

$$E_r n_z - E_z n_r = 0, \quad E_\varphi = 0, \tag{16}$$

where $\mathbf{n} = (n_r, 0, n_z)$ is the inner unit normal to the boundary of the flow region.

Taking into account $\mathbf{vn} = 0$ and $\mathbf{Hn} = 0$, from (13)-(16) we obtain

$$\frac{\partial H_\varphi}{\partial n} + \frac{H_\varphi}{r} n_r = \frac{\partial}{\partial n}(r H_\varphi) = 0; \tag{17}$$

$$\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} = \frac{1}{r}\left(\frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r}\frac{\partial \Psi}{\partial r} + \frac{\partial^2 \Psi}{\partial z^2}\right) = 0. \tag{18}$$

It follows from the condition $\mathbf{hn} = 0$ and Eqs. (10) and (18) that Ψ takes constant values on the boundaries. In the general case, these values are different on different boundaries for a multiply connected region. If this region is simply connected, one can set $\Psi = 0$ on the boundary without loss of generality.

Thus, solutions of system (10)-(12) should satisfy the following boundary conditions on boundary l (boundaries) of the flow region:

$$\Psi = \text{const}, \quad \partial\gamma/\partial n = 0. \tag{19}$$

The following conditions should be satisfied for Γ: the condition of attachment $\Gamma = 0$ on a part (parts) of boundary l', and the condition of the absence of tangential stresses [7] on the other part (parts) l'':

$$\Gamma = 0 \quad \text{on} \quad l', \qquad \frac{\partial \Gamma}{\partial n} - \frac{2\Gamma}{r} n_r = 0 \quad \text{on} \quad l''. \tag{20}$$

If the axis of symmetry (the z axis) belongs to the flow region, we have $\Psi = \Gamma = \gamma = 0$ on this axis, and $\Psi \simeq \Gamma \simeq \gamma \simeq r^2$.

We consider the case of a simply connected flow region. Let Ψ, Γ, and γ for $t = 0$. Multiplying (10) by $r\Psi$, we have

$$\begin{aligned} &\frac{\partial}{\partial t}\frac{r}{2}\Psi^2 + \frac{\partial}{\partial r}\left(\frac{r}{2}u\Psi^2\right) + \frac{\partial}{\partial z}\left(\frac{r}{2}u\Psi^2\right) \\ &= \nu_m\left[\frac{\partial}{\partial r} r\Psi\frac{\partial \Psi}{\partial r} + \frac{\partial}{\partial z} r\Psi\frac{\partial \Psi}{\partial z} - \frac{\partial}{\partial r}\frac{1}{2}\Psi^2 - r\left[\left(\frac{\partial \Psi}{\partial r}\right)^2 + \left(\frac{\partial \Psi}{\partial z}\right)^2\right]\right]. \end{aligned} \tag{21}$$

Integrating (21) over meridional section D and taking (19) into account, we obtain

$$\begin{aligned} \frac{\partial}{\partial t}\int_D \frac{r}{2}\Psi^2 drdz = &-\oint_l \frac{r}{2}\Psi^2(\mathbf{vn})dl + \nu_m \oint_l \Psi^2 n_r dl \\ &+ \nu_m \oint_l r\Psi\frac{\partial \Psi}{\partial n} dl - \nu_m \int_D r\left[\left(\frac{\partial \Psi}{\partial r}\right)^2 + \left(\frac{\partial \Psi}{\partial z}\right)^2\right] drdz. \end{aligned} \tag{22}$$

Taking into account the boundary condition $\Psi = 0$ on boundary l, from (22) we find that

$$\frac{\partial}{\partial t}\int_D \frac{r}{2}\Psi^2 drdz = -\nu_m \int_D r\left[\left(\frac{\partial \Psi}{\partial r}\right)^2 + \left(\frac{\partial \Psi}{\partial z}\right)^2\right] drdz, \tag{23}$$

and, hence, $\Psi \to 0$ for $t \to \infty$ if $\nu_m \neq 0$, i.e., the fluid has finite conductivity. By virtue of this, for sufficiently large t, the second term on the right side of the equation for Γ (11) becomes negligibly small, and, according tò [7], $\Gamma \to 0$ if attachment conditions (20) are fulfilled on an infinitesimal but finite part of boundary l'. Consequently, axisymmetric spontaneous swirling in the flow region with a simply connected meridional section is impossible.

The behavior of the azimuthal component of the magnetic field for fairly large t is governed by Eq. (12), in which terms containing Ψ and Γ are omitted. This equation is written as

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial r}uh + \frac{\partial}{\partial z}wh = \nu_m\left[\frac{\partial}{\partial r}\left(\frac{\partial h}{\partial r} + \frac{h}{r}\right) + \frac{\partial^2 h}{\partial z^2}\right]. \tag{24}$$

Integrating (24) over the section D, we have

$$\frac{\partial}{\partial t}\int\limits_D h\,dr\,dz = -\oint\limits_l (\mathbf{vn})h\,dl + \nu_m \oint\limits_l \left(\frac{\partial h}{\partial n} + \frac{h}{r}n_r\right)dl. \tag{25}$$

Let the axis of symmetry be contained in the flow region and, hence, be part of the boundary of region D. By virtue of the boundary conditions, the right side is equal to zero everywhere on the boundary, except in the region coinciding with the axis of symmetry, in which the second term in (25) is different from zero. If the values of h are positive (negative) everywhere inside region D, the right side of (25) is nonpositive (nonnegative), since $\partial h/\partial n \leqslant 0$ and $n_r < 0$ on the axis and h decreases. If h changes sign in region D, there is a curve $h = 0$ with a time-dependent location inside region D. In this case, integration is performed over subregion D', in which the values of h are positive (negative). By virtue of the fact that on the additional boundary h vanishes, for subregion D' we obtain an equality similar to (25), from which follows the conclusion that the magnetic field decreases in this subregion, and, hence, over the entire region.

If the flow region does not include the axis of symmetry, i.e., it is toroidal, by virtue of the boundary conditions, from (25) follows the well-known law of conservation of an azimuthal magnetic field flux over section D. Proceeding as above, one can see that in this case $h \to 0$ as $t \to \infty$ if the flux was initially equal to zero. If the initial flux is different from zero, h does not vanish, and the evolution of the azimuthal component of the magnetic field is determined by the poloidal velocity components u and w.

Thus, axisymmetric spontaneous swirling is impossible if the meridional section of the flow region is simply connected.

We now consider the case of a multiply connected section D, where one or several toroidal superconducting bodies are located inside the main flow region so that the axial symmetry is conserved. The constants Ψ_i can take different values on the inner boundaries l_i. If all Ψ_i are equal to zero, the conclusion that axisymmetric spontaneous swirling is impossible remains valid for the multiply connected section, since the poloidal magnetic field vanishes as for a simply connected section.

If Ψ_i are different from zero (at least, on the inner boundary), the poloidal magnetic field does not disappear. If, in this case, $h \not\equiv 0$ ($\gamma \neq \text{const}$), the electromagnetic forces, according to Eq. (11), necessarily generate a certain swirling flow ($\Gamma \not\equiv 0$). As was mentioned above, the case $h \not\equiv 0$ takes place if the main flow region does not include the axis of symmetry (is toroidal) and the azimuthal flow is different from zero. In the problem considered, such flows are not interesting and should be eliminated.

Let there be an axisymmetric flow with $\Gamma = 0$ and $h = 0$ or $h = \gamma_0/r$. The second case is possible if the fluid in a cavity is at rest. For $t = 0$, the perturbation $\Gamma \neq 0$ and $h = 0$ is assumed. If, then, $\Gamma \to 0$ for $t \to \infty$, spontaneous swirling is impossible. In this formulation the question of spontaneous swirling for a multiply connected flow remains open. For an unbounded flow region, simple connectedness does not suggest the disappearance of the poloidal magnetic-field components if one does not require that they should vanish at infinity. As an example of an MHD flow of this type, we consider a Burgers vortex and its plane analog in a conductive incompressible viscous fluid.

We consider first the plane analog. Equations (1)–(3) have solutions of the form

$$u = ax, \quad v = -ay, \quad w = w(t,x,y), \quad h_x = bx, \quad h_y = -by,$$
$$h_z = h(t,x,y), \quad P = P_0 - (1/2)\rho a^2(x^2+y^2) - (1/2)\rho h^2,$$

where a and b are positive constants, and the transverse velocity and magnetic-field components w and h satisfy the system

$$\frac{\partial w}{\partial t} + ax\frac{\partial w}{\partial x} - ay\frac{\partial w}{\partial y} = bx\frac{\partial h}{\partial x} - by\frac{\partial h}{\partial y} + \nu\Delta w; \tag{26}$$

$$\frac{\partial h}{\partial t} + ax\frac{\partial h}{\partial x} - ay\frac{\partial h}{\partial y} = bx\frac{\partial w}{\partial x} - by\frac{\partial w}{\partial y} + \nu_m\Delta h. \tag{27}$$

Let $\nu = \nu_m = 0$, $w = w(t,y)$, and $h = h(t,y)$. Then, the general solution of system (26) and (27) has the form

$$w = f(ye^{(a-b)t}) + \Psi(ye^{(a+b)t}); \tag{28}$$

$$h = f(ye^{(a-b)t}) - \Psi(ye^{(a+b)t}) \tag{29}$$

where f and Ψ are arbitrary functions.

We consider the field momenta w and h. Directly from solutions (28) and (29) or from Eqs. (26) and (27) we find that the quantities

$$A_n = \int_{-\infty}^{\infty} y^n w dy, \quad B_n = \int_{-\infty}^{\infty} y^n h dy$$

satisfy the equations

$$dA/dt = (n+1)(bB - aA), \quad dB/dt = (n+1)(bA - aB). \tag{30}$$

Taking into account that $A_n = A_{0n}$ and $B_n = 0$ for $t = 0$, from (30) we obtain

$$A_n = A_{0n}e^{-(n+1)at}\cosh(n+1)bt, \quad B_n = A_{0n}e^{-(n+1)at}\sinh(n+1)bt. \tag{31}$$

Similarly we find the energy

$$\varepsilon = \int_{-\infty}^{\infty} \left(\frac{1}{2}w^2 + \frac{1}{2}h^2\right) dy = \varepsilon_0 e^{-at}\cosh bt.$$

This means that all A_n, B_n, and ε, including the transverse momentum A_0, increase with time if $b > a$ for $n \geqslant 0$. Relations (31) remain valid for A_0 and B_0 and for a viscous fluid with finite conductivity, and, hence, the transverse momentum also increases with time for a viscous fluid with finite conductivity. In this formulation, however, if $b > a$, the transverse momentum increases on account of the transverse-momentum flux $\Pi_{xz} = -h_z h_x = -bxh$, which is related to the Maxwell stress tensor of a magnetic field that flows from infinity along the x axis to the plane $x = 0$. Therefore, the flow considered is not an example of the spontaneous occurrence of transverse flow.

Note also that, although the quantities A_0 and B_0 increase with time, the quantities w and h tend to zero as $t \to \infty$. This is shown as follows. The functions w and h are represented as the sum of the even and uneven terms, so that

$$w = w_1 + w_2, \quad h = h_1 + h_2, \quad w_1(-y) = -w_1(y), \quad w_2(-y) = w_2(y),$$
$$h_1(-y) = -h_1(y), \quad h_2(-y) = h_2(y).$$

Assuming that $w_1 = y\Omega$ and $h_1 = y\omega$ for Ω and ω, we have

$$\Omega_t - a\frac{\partial}{\partial y}(y\Omega) = -b\frac{\partial}{\partial y}(y\omega) + \frac{\nu}{y}\frac{\partial^2}{\partial y^2}(y\Omega); \tag{32}$$

$$\omega_t - a\frac{\partial}{\partial y}(y\omega) = -b\frac{\partial}{\partial y}(y\Omega) + \frac{\nu_m}{y}\frac{\partial^2}{\partial y^2}(y\omega). \tag{33}$$

Multiplying (32) by $y\Omega$ and (33) by $y\omega$ and integrating with respect to y, we obtain

$$\frac{d}{dt}\int_0^{\infty}\frac{1}{2}y(\Omega^2+\omega^2)dy = -\nu\int_0^{\infty} y\left(\frac{\partial\omega}{\partial y}\right)^2 dy - \nu_m\int_0^{\infty} y\left(\frac{\partial\Omega}{\partial y}\right)^2 dy - \frac{\nu}{2}\Omega^2(0) - \frac{\nu_m}{2}\omega^2(0). \tag{34}$$

For even constituents, we set $p = \partial w_2/\partial y$ and $q = \partial h_2/\partial y$. The equations for p and q have the form

$$p_t - a\frac{\partial}{\partial y}(yp) = -b\frac{\partial}{\partial y}(yq) + \nu p_{yy}; \tag{35}$$

$$q_t - a\frac{\partial}{\partial y}(yq) = -b\frac{\partial}{\partial y}(yp) + \nu_m q_{yy}. \tag{36}$$

Multiplying (35) by yp and (36) by yq and taking into account that $p = q = 0$ for $y = 0$, by virtue of the evenness of w_2 and h_2, we obtain

$$\frac{d}{dt}\int_0^\infty \frac{1}{2} y(p^2+q^2)dy = -\nu\int_0^\infty y\Big(\frac{\partial p}{\partial y}\Big)^2 dy - \nu_m \int_0^\infty y\Big(\frac{\partial q}{\partial y}\Big)^2 dy. \tag{37}$$

From (34) and (37) it follows that $w \to 0$ as $t \to \infty$.

For $t = 0$, we now specify $w = w_0(x, y)$ and $h = h_0(x, y)$, which vanish at infinity. Then, multiplying Eq. (26) by w and Eq. (27) by h, we obtain

$$\frac{d}{dt}\iint_{-\infty}^{\infty} \frac{1}{2}(w^2+h^2)dxdy = -\nu\iint_{-\infty}^{\infty}\Big[\Big(\frac{\partial w}{\partial x}\Big)^2 + \Big(\frac{\partial w}{\partial y}\Big)^2\Big]dxdy - \nu_m\iint_{-\infty}^{\infty}\Big[\Big(\frac{\partial h}{\partial x}\Big)^2 + \Big(\frac{\partial h}{\partial y}\Big)^2\Big]dxdy. \tag{38}$$

The transverse momentum $\iint_{-\infty}^{\infty} w\,dx\,dy$ and the magnetic-field flux $\iint_{-\infty}^{\infty} h\,dx\,dy$ are conserved. Thus, it follows from (38) that spontaneous transverse flow does not arise in this case.

We now consider a magnetohydrodynamic analog of the Burgers vortex. System (1)–(3) has solutions of the form

$$u = -ar, \quad w = 2az, \quad h_r = -br, \quad h_z = 2bz; \tag{39}$$

$$\frac{1}{\rho}\frac{\partial P}{\partial r} = -a^2 r - hh_r + \frac{1}{r}(v^2 - h^2), \quad \frac{1}{\rho}\frac{\partial P}{\partial z} = -4a^2 z - hh_z, \tag{40}$$

where a and b are positive constants, and Eqs. (40) are integrable only on the condition that the azimuthal velocity $v_\varphi = v(r,t)$ and magnetic-field $h_\varphi = h(r,t)$ components are independent of z. According to this, v and h satisfy the equations

$$\frac{\partial v}{\partial t} - ar\frac{\partial v}{\partial r} - av = -br\frac{\partial h}{\partial r} - bh + \nu\Big(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\Big); \tag{41}$$

$$\frac{\partial h}{\partial t} - ar\frac{\partial h}{\partial r} + ah = -br\frac{\partial v}{\partial r} + bv + \nu_m\Big(\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} - \frac{h}{r^2}\Big). \tag{42}$$

We consider at first the case of a nonviscous, ideally conducting fluid ($\nu = 0$ and $\nu_m = 0$). Let for $t = 0$ a perturbation $v = v_0(r)$ and $h = 0$ be introduced such that the quantity

$$M_0 = \int_{-\infty}^{\infty} |v_0(x)|dx, \quad x = \ln(r/r_0) \quad (r_0 = \text{const})$$

has a finite value.

The solution of system (41) and (42) subject to this initial condition is written as

$$v = \int_{-\infty}^{\infty} A(k)e^{ik(x+at)}\Big(\cos\lambda(k)t + \frac{a}{\lambda(k)}\sin\lambda(k)t\Big)dk; \tag{43}$$

$$h = \int_{-\infty}^{\infty} \frac{b(1-ik)}{\lambda(k)} A(k)e^{ik(x+at)}\sin\lambda(k)t\,dk. \tag{44}$$

Here

$$\lambda(k) = (b^2 - a^2 + b^2k^2)^{1/2}; \quad A(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} v_0(x)e^{-ikx}dx. \tag{45}$$

For the field momenta v and h we have

$$A_n(t) = \int_0^\infty r^n v dr, \quad B_n(t) = \int_0^\infty r^n h dr.$$

Multiplying (41) and (42) by r^n and integrating for $\nu = \nu_m = 0$, we obtain

$$dA_n/dt + naA_n = nbB_n, \quad dB_n/dt + (n+2)aB_n = (n+2)bA_n. \tag{46}$$

Taking into account that $A_n(0) = A_{n0}$ and $B_n(0) = 0$ for $t = 0$, from (46) we find that

$$A_n = A_{n0}e^{-(n+1)at}\Big(\cosh\mu_n t + \frac{a}{\mu_n}\sinh\mu_n t\Big); \tag{47}$$

$$B_n = \frac{(n+2)b}{\mu_n}A_{n0}e^{-(n+1)at}\sinh\mu_n t, \tag{48}$$

where

$$\mu_n = (a^2 + n(n+2)b^2)^{1/2}. \tag{49}$$

From the solution of system (43) and (44), it follows that for $a > b$ (a weak poloidal field) the azimuthal velocity component v increases infinitely, but all the momenta $A_n \to 0$ if $n \geqslant 0$. In the case of $n = -1$ ($a > b$), we have

$$A_{-1} = A_{-10}\Big(\cosh(a^2 - b^2)^{1/2}t + \frac{a}{(a^2 - b^2)^{1/2}}\sinh(a^2 - b^2)^{1/2}t\Big).$$

The unbounded growth in the momentum A_{-1} indicates that $v(r,t)$ increases near the axis. In this case, the angular momentum (per unit length) $A_2 \to 0$, and the energy (per unit length) is bounded:

$$\int_0^\infty \frac{1}{2}r(h^2 + v^2)dr \leqslant \frac{a}{a-b}\int_0^\infty \frac{1}{2}rv_0^2dr.$$

For nonzero viscosity this growth ceases, and $v \to 0$, as can be proved rigorously for $b = 0$. For $b \neq 0$, we were unable to obtain a rigorous proof.

For $b = a$, the general solution of system (41) and (42) has the form ($\nu = \nu_m = 0$)

$$v = F(r) + f(\xi) + \xi f'(\xi), \quad h = F(r) + f(\xi) - \xi f'(\xi). \tag{50}$$

Here $\xi = r\exp(2at)$ and F and f are arbitrary functions. In this case, as $t \to \infty$, we have $A_n \to B_n \to (n+2)A_{0n}/[2(n+1)]$ for $n \geqslant 0$, and $A_{-1} = A_{-10}at$, and the energy is bounded, which can be shown directly from (50).

For $b > a$ (a strong poloidal magnetic field), all A_n for $n \geqslant 1$ increase with time (including the angular momentum A_2). For $n = 0$, the azimuthal rate A_0 and the azimuthal magnetic-field flux B_0 remain bounded. The time dependence of the momenta A_{-1} and B_{-1} for which $\mu_{-1} = i(b^2 - a^2)^{1/2}$ has the form

$$A_{-1} = A_{-10}\Big(\cos(b^2 - a^2)^{1/2}t + \frac{a}{(b^2 - a^2)^{1/2}}\sin(b^2 - a^2)^{1/2}t\Big),$$

$$B_{-1} = \frac{bA_{-10}}{(b^2 - a^2)^{1/2}}\sin(b^2 - a^2)^{1/2}t,$$

which indicates that the resulting flow is oscillating, at least, near the symmetry axis.

Relations (46), and, hence, (47) and (48), remain valid for $n = 2$ for a viscous fluid with finite conductivity. Thus, in this case, swirling occurs for $b > a$. However, as for the plane analog, the growth in the angular momentum is related to the angular-momentum flux through the Maxwell stress tensor of the magnetic field at infinity, and the swirling is not spontaneous.

The above example, although somewhat artificial, shows that the formulation of problems of spontaneous swirling in magnetohydrodynamic flows should ensure control (absence) of not only the kinematical flux of the mechanical angular momentum, but also of the angular momentum flux related to the stress tensor of the electromagnetic field flowing from infinity into the flow region.

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